ON THE PROPAGATION OF ELASTIC WAVES IN TWO-PHASE MEDIA

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At the present time a number of papers has been already devoted to the dynamics of two-phase media. One may mention the papers by Frenkel' [1], Rakhmatulin [2], Biot [3,4], Zwikker and Kosten [5], and others. However, the basic problem of the setting up of the equations of motion in two-phase media still cannot be considered solved and requires additional study and experimental verification.

This paper is concerned with the study of the simplest case of motion, which is the propagation of elastic waves in a homogeneous isotropic medium consisting of a solid and a fluid phase. The problems of the reflection of plane waves and surface waves at the free boundary of the half-space are solved. It is shown that the stress-strain relations established by Frenkel' are equivalent to the analogous relations proposed by Biot and that the equations of motion of the latter are more general.

1. Basic equations. The interpenetrating motion of the solid and fluid phases will be analysed as a motion of the fluid in a deforming porous medium. We shall assume that the dimensions of the pores are small as compared to the distance in which the kinematic and dynamic characteristics of motion change substantially. This allows us to assume both media to be continuous, and thus we shall have at every point of the space two displacement vectors: the displacement vector u of the solid phase (the skeleton of the porous medium) and the displacement vector v of the fluid. As it was shown by Frenkel' [1], the total stress tensor in the skeleton (taking into account the pressure of the fluid in the pores) can be written in the form

$$P_{ik} = L\theta\delta_{ik} + 2Gu_{ik} + R_0\left(1 - m - \frac{K}{K_0}\right)\varphi\delta_{ik} \qquad (1.1)$$

$$K = L + \tfrac{2}{3} G, \qquad u_{ik} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$\theta = u_{11} + u_{22} + u_{33} = \operatorname{div} u, \qquad \varphi = -\frac{\Delta \rho}{\rho}$$

where L, G, and K are respectively the Lamé coefficients and the bulk modulus of the porous skeleton with empty pores, K_0 is the actual bulk modulus of the solid phase, R_0 is the bulk modulus of the fluid, m is the porosity, u_{ij} is the strain tensor of the skeleton, θ is the relative volume change (dilatation) of the skeleton, ϕ is the relative change of the actual density of the fluid, δ_{ik} is the Kronecker delta, which is equal to unity when $i = k$ and equal to zero when $i \neq k$.

The change of the porosity can be expressed according to [1] by the formula

$$\Delta m = a\left(\theta - \frac{R_0}{K_0}\varphi\right) \tag{1.2}$$

The constant coefficient a is some additional parameter which characterizes the elastic properties of the porous medium (when the porosity is constant $a = 0$).

A small change of pressure in the liquid is related to a small change of density by the equation

$$\Delta p = -R_0 \varphi \tag{1.3}$$

The continuity equation expressing the law of conservation of mass has the form

$$\frac{\partial}{\partial t}(m\rho) + \operatorname{div}\left(m\rho \frac{\partial v}{\partial t}\right) = 0 \tag{1.4}$$

By linearizing and expressing it in finite differences we obtain

$$m\Delta\rho + \rho\Delta m + m\rho\varepsilon = 0 \qquad \varepsilon = \operatorname{div} v \tag{1.5}$$

where ϵ is the relative change of the volume (dilatation) of the fluid. By taking into account (1.2) we find

$$\varphi = \beta\left(\frac{a}{m}\theta + \varepsilon\right), \qquad \beta = \frac{1}{1 + aR_0 / mK_0} \tag{1.6}$$

When (1.6) is substituted into (1.1) and (1.3) they become

$$P_{ik} = \lambda\theta\delta_{ik} + 2\mu u_{ik} + Q\varepsilon\delta_{ik}, \qquad \sigma = -m\Delta p = Q'\theta + R\varepsilon \tag{1.7}$$

where σ represents the force acting upon the fluid per unit cross section of the porous medium, and the other symbols are

$$\lambda = L + \beta \frac{a}{m}\left(1 - m - \frac{K}{K_0}\right) R_0, \qquad \mu = G$$

$$Q = \beta\left(1 - m - \frac{K}{K_0}\right) R_0, \qquad Q' = \beta a R_0, \qquad R = \beta m R_0 \tag{1.8}$$

The work of the forces acting upon an element of the solid and fluid phase is

$$d'A = P_{ik}\, du_{ik} + \sigma d\varepsilon = \lambda \theta d\theta + 2\mu u_{ik}\, du_{ik} + Q\varepsilon d\theta + Q'\theta d\varepsilon + R\varepsilon d\varepsilon$$

For the existence of potential energy this expression should be a total differential, and consequently the relation $Q = Q'$ holds and thus, when (1.8) is utilized, we obtain

$$a = 1 - m - \frac{K}{K_0} \tag{1.9}$$

Equation (1.7) coincides with the stress-strain relations established by Biot [3]. If the shear in the skeleton is neglected the stress tensor P_{ik} will contain only diagonal terms, i.e. $P_{ik} = -p_1\delta_{ik}$, where p_1 is the pressure force on the skeleton per unit cross-sectional area of the porous medium. Equations (1.7) then become:

$$-p_1 = \left(\lambda + \frac{2}{3}\mu\right)\theta + Q\varepsilon, \qquad -m\Delta p = Q\theta + R\varepsilon \tag{1.10}$$

Let $R_0 \ll K \ll K_0$ (this can happen in the case when the pores are filled with a gas). In accordance with (1.8) and (1.2) we obtain for this case

$$\lambda + \frac{2}{3}\mu = K, \quad R = mR_0, \quad Q = (1-m) R_0, \Delta m = (1-m)\theta \tag{1.11}$$

We introduce the quantity

$$p_2 = mp - m_0 p_0 \tag{1.12}$$

Here m_0 and p_0 are the equilibrium values of the porosity and the pressure of the fluid (gas). Using the equation

$$\frac{\partial p_2}{\partial t} = m\frac{\partial p}{\partial t} + p_0 \frac{\partial m}{\partial t}$$

and taking into account (1.10) and (1.11) we obtain (1.13)

$$-\frac{\partial p_1}{\partial t} = K\frac{\partial \theta}{\partial t} - \frac{1-m}{m}\frac{\partial p_2}{\partial t}, \qquad -\frac{\partial p_2}{\partial t} = mR_0\frac{\partial \varepsilon}{\partial t} + (1-m)(R_0 - p_0)\frac{\partial \theta}{\partial t}$$

which coincides with the equations of Zwikker and Kosten [5].

Let us now obtain the equations of motion, at first neglecting the viscosity of the fluid.

The force acting on the skeleton (per unit volume of the porous medium) is equal to the time derivative of the total momentum of the skeleton (in this volume). The latter consists of the momentum of the skeleton with absolutely empty pores and the additional momentum arising due to the relative motion of the phases of the medium, i.e.

$$j_1 = \rho_1 \frac{\partial u}{\partial t} + \rho_{12}\left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\right) = \rho_{11} \frac{\partial u}{\partial t} + \rho_{12} \frac{\partial v}{\partial t} \tag{1.14}$$

Thus we have

$$\rho_{11} \frac{\partial^2 u_i}{\partial t^2} + \rho_{12} \frac{\partial^2 v_i}{\partial t^2} = \frac{\partial P_{ik}}{\partial x_k} \tag{1.15}$$

Analogously we find the total momentum of the fluid

$$j_2 = \rho_2 \frac{\partial v}{\partial t} + \rho_{12}\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right) = \rho_{12} \frac{\partial u}{\partial t} = \rho_{22} \frac{\partial v}{\partial t} \tag{1.16}$$

and obtain the second equation of motion

$$\rho_{12} \frac{\partial^2 u_i}{\partial t^2} + \rho_{22} \frac{\partial^2 v_i}{\partial t^2} = \frac{\partial \sigma}{\partial x_i} \tag{1.17}$$

The coefficient ρ_{12} shall be called, according to [4], the coefficient of dynamic coupling of the skeleton with the fluid, ρ_1 and ρ_2 are the masses of the solid and fluid phases per unit volume of the medium, ρ_{11} and ρ_{22} can be regarded as some "effective masses" of these phases:

$$\rho_{11} = \rho_1 - \rho_{12}, \qquad \rho_{22} = \rho_2 - \rho_{12} \tag{1.18}$$

Since during the motion of a solid body in a fluid its "effective mass" is larger than its actual mass ($\rho_{11} > \rho_1$), the coefficient ρ_{12} has to be negative.

In order to take into account the viscosity in equations (1.15) and (1.17), one has to add a term that is proportional to the difference between the velocities of the two phases:

$$\begin{aligned} \rho_{11} \frac{\partial^2 u_i}{\partial t^2} + \rho_{12} \frac{\partial^2 v_i}{\partial t^2} + b\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) &= \frac{\partial P_{ik}}{\partial x_k} \\ \rho_{12} \frac{\partial^2 u_i}{\partial t^2} + \rho_{22} \frac{\partial^2 v_i}{\partial t^2} + b\left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t}\right) &= -m \frac{\partial p}{\partial x_i} \end{aligned} \tag{1.19}$$

The constant b can be determined from the condition of satisfying the Darcy law for the particular case of stationary flow of the fluid. Thus we find

$$b = \frac{\mu m^2}{k} \tag{1.20}$$

where μ is the coefficient of viscosity and k is the coefficient of permeability, which is proportional to the porosity and the square of the pore diameters:

$$k = \text{const}\, m\delta^2 \tag{1.21}$$

If the coefficient of dynamic coupling ρ_{12} is neglected in (1.19), we obtain the equations of motion introduced by Frenkel' [1]. In the case of harmonic waves of frequency ω (where the displacement vectors depend on time as $e^{-i\omega t}$, equations (1.19) can also be written in the following form

$$\rho_1 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial P_{ik}}{\partial x_k} + s\left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t}\right), \qquad \rho_2 \frac{\partial^2 v_i}{\partial t^2} = -m\frac{\partial p}{\partial x_i} + s\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{1.22}$$

where

$$s = b - i\omega\rho_{12} \tag{1.23}$$

If the shear in the skeleton is neglected, i.e.

$$P_{ik} = -p_1\delta_{ik} = -(1-m)p_s\delta_{ik}$$

(P_s is the actual pressure in the skeleton), we obtain from (1.22)

$$\frac{\partial^2 u_i}{\partial t^2} = -\frac{1}{\rho_s}\frac{\partial P_s}{\partial x_i} + K_{12}\left(\frac{\partial v_i}{\partial t} - \frac{\partial u_i}{\partial t}\right), \qquad \frac{\partial^2 v_i}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + K_{21}\left(\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial t}\right) \tag{1.24}$$

Here ρ_s and ρ are the actual densities of the solid and fluid phases:

$$\rho_s = \frac{\rho_1}{1-m}, \qquad \rho = \frac{\rho_2}{m}, \qquad K_{12} = \frac{s}{\rho_1}, \qquad K_{21} = \frac{s}{\rho_2} \tag{1.25}$$

Equations (1.24) coincide with the equations of motion introduced by Rakhmatulin [2] for multi-phase liquid or gaseous media.

Let us decompose the displacement vectors into irrotational and solenoidal components

$$\begin{aligned} u &= u_l + u_t, \qquad \operatorname{rot} u_l = 0, \qquad \operatorname{div} u_t = 0 \\ v &= v_l + v_t, \qquad \operatorname{rot} v_l = 0, \qquad \operatorname{div} v_t = 0 \end{aligned} \tag{1.26}$$

Equations (1.19), together with (1.7), (1.9), and (1.26) reduce to the following system:

$$\begin{aligned} \rho_{11}\frac{\partial^2 u_l}{\partial t^2} + \rho_{12}\frac{\partial^2 v_l}{\partial t^2} + b\frac{\partial}{\partial t}(u_l - v_l) &= (\lambda + 2\mu)\nabla^2 u_l + Q\nabla^2 v_l \\ \rho_{12}\frac{\partial^2 u_l}{\partial t^2} + \rho_{22}\frac{\partial^2 v_l}{\partial t^2} + b\frac{\partial}{\partial t}(v_l - u_l) &= Q\nabla^2 u_l + R\nabla^2 v_l \\ \rho_{11}\frac{\partial^2 u_t}{\partial t^2} + \rho_{12}\frac{\partial^2 v_t}{\partial t^2} + b\frac{\partial}{\partial t}(u_t - v_t) &= \mu\nabla^2 u_t \\ \rho_{12}\frac{\partial^2 u_t}{\partial t^2} + \rho_{22}\frac{\partial^2 v_t}{\partial t^2} + b\frac{\partial}{\partial t}(v_t - u_t) &= 0 \end{aligned} \tag{1.27}$$

In the case of monochromatic waves of frequency ω, the first two equations of (1.27) with the aid of the linear transformation

$$u_l = u_1 + u_2, \quad v_l = M_1 u_1 + M_2 u_2 \tag{1.28}$$

and after the introduction of the following notation

$$\gamma_{11} = \frac{\rho_{11}}{\rho}, \quad \gamma_{12} = \frac{\rho_{12}}{\rho}, \quad \gamma_{22} = \frac{\rho_{22}}{\rho}, \quad \rho = \rho_{11} + \rho_{22} + 2\rho_{12}$$
$$\sigma_{11} = \frac{\lambda + 2\mu}{H}, \quad \sigma_{12} = \frac{Q}{H}, \quad \sigma_{22} = \frac{R}{H}, \quad H = \lambda + 2\mu + R + 2Q \tag{1.29}$$
$$c^2 = \frac{H}{\rho}$$

reduce to

$$\nabla^2 u_1 + k_1^2 u_1 = 0, \qquad \nabla^2 u_2 + k_2^2 u_2 = 0 \tag{1.30}$$

where

$$k_1^2 = z_1 \left(\frac{\omega}{c}\right)^2, \qquad k_2^2 = z_2 \left(\frac{\omega}{c}\right)^2 \tag{1.31}$$

z_1 and z_2 are roots of the quadratic equation

$$(\sigma_{11}\sigma_{22} - \sigma_{12}^2) z^2 - (\sigma_{11}\gamma_{22} + \sigma_{22}\gamma_{11} - 2\sigma_{12}\gamma_{12}) z +$$
$$+ \gamma_{11}\gamma_{22} - \gamma_{12}^2 - \frac{ib}{\omega\rho}(z - 1) = 0 \tag{1.32}$$

and the coefficients of the transformation (1.28) are determined from the formulas

$$M_1 = \frac{-\gamma_{12} + z_1\sigma_{12} + i\gamma}{\gamma_{22} - z_1\sigma_{22} + i\gamma}, \quad M_2 = \frac{-\gamma_{12} + z_2\sigma_{12} + i\gamma}{\gamma_{22} - z_2\sigma_{22} + i\gamma}, \quad \gamma = \frac{b}{\rho\omega} \tag{1.33}$$

Equations (1.30) describe the propagation of longitudinal waves of type I and II.

If the coupling between the solid and the fluid phases of the medium is weak

$$\gamma_{12} = 0, \quad \sigma_{12} = 0, \quad \gamma = 0,$$

then we find from (1.32)

$$z_1 = \frac{\gamma_{11}}{\sigma_{11}}, \qquad z_2 = \frac{\gamma_{22}}{\sigma_{22}} \tag{1.34}$$

and in addition to that the velocities of the I and II longitudinal waves are obviously equal to the wave velocities propagating separately in continuous elastic and continuous fluid media. According to (1.33) we have in this case

$$M_1 = 0, \qquad M_2 = \infty \tag{1.35}$$

Let us find the approximate values of the roots of equation (1.32) with the condition $\gamma \gg 1$, which corresponds to the case of low frequency or high viscosity

$$z_1 = 1 - \frac{i}{\gamma}(\sigma_{11}\sigma_{22} - \sigma_{12}^2 - \sigma_{11}\gamma_{22} - \sigma_{22}\gamma_{11} + 2\sigma_{12}\gamma_{12} + \gamma_{11}\gamma_{22} - \gamma_{12}^2)$$
$$z_2 = i\frac{\gamma}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{1.36}$$

As can be easily verified, the damping coefficients of the wave (being the imaginary parts of k_1 and k_2) will be here proportional to the square of the frequency for the I longitudinal wave and to the square root of the frequency for the II longitudinal wave. Thus for a very low frequency the I longitudinal wave experiences a negligible damping, while the II longitudinal wave practically disappears because of the large damping.

The second two equations of (1.27), which describe the propagation of the transverse wave, reduce to

$$v_t = M_t u_t, \qquad \nabla^2 u_t + k_t^2 u_t = 0 \tag{1.37}$$

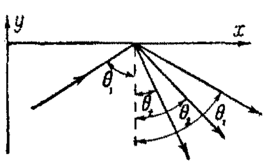
where

$$M_t = \frac{-\gamma_{12} + i\gamma}{\gamma_{22} + i\gamma}, \qquad k_t^2 = \frac{\rho_1 + \rho_2 M_t}{\mu}\omega^2 \tag{1.38}$$

With a low frequency the damping coefficient is here also proportional to the square of the frequency just as in the case of the I longitudinal wave.

The existence of three types of waves in porous media and also their damping characteristics were established in the papers [1] and [4]. If $\gamma \ll 1$, one can neglect the effect of viscosity. It is necessary here, however, to keep in mind the fact that ω should remain smaller than the frequency at which the wave length is comparable to the dimensions of the pores.

2. Reflection of plane waves at the free boundary of the half-space. The boundary conditions at the free surface of the medium will be



$$P_{yy} = 0, \qquad P_{xy} = 0, \qquad \sigma = 0 \tag{2.1}$$

or on the basis of (1.7)

$$\lambda\theta + 2\mu u_{yy} + Q\varepsilon = 0$$
$$u_{xy} = 0 \tag{2.2}$$
$$Q\theta + R\varepsilon = 0$$

After introducing the potentials of the longitudinal and transverse waves and taking into account (1.26), (1.28) and (1.37), we can write the expressions for the total displacement vectors of the skeleton and the fluid

$$u = \text{grad}\varphi_1 + \text{grad}\varphi_2 + \text{rot}\Psi,\ v = M_1 \text{grad}\varphi_1 + M_2 \text{grad}\varphi_2 + M_t \text{rot}\Psi \quad (2.3)$$

after which the boundary conditions reduce to

$$(\lambda + QM_1)\nabla^2\varphi_1 + (\lambda + QM_2)\nabla^2\varphi_2 + 2\mu\left(\frac{\partial^2\varphi_1}{\partial y^2} + \frac{\partial^2\varphi_2}{\partial y^2} + \frac{\partial^2\Psi}{\partial x \partial y}\right) = 0$$

$$2\left(\frac{\partial^2\varphi_1}{\partial x \partial y} + \frac{\partial^2\varphi_2}{\partial x \partial y}\right) + \frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial y^2} = 0 \quad (2.4)$$

$$(Q + RM_1)\nabla^2\varphi_1 + (Q + RM_2)\nabla^2\varphi_2 = 0$$

We shall study the case of the incidence of a I longitudinal wave on the boundary. The three boundary conditions can be satisfied only be allowing the incident wave to create a system of three reflected waves at the boundary: two longitudinal waves and a transverse wave. Thus the complete sonic field in the medium will be (see Fig.)

$$\varphi_1 = Ae^{ik_1(x \sin\theta_1 + y\cos\theta_1)} + AW_1 e^{ik_1(x\sin\theta_1 - y\cos\theta_1)}$$

$$\varphi_2 = AW_2 e^{ik_2(x\sin\theta_2 - y\cos\theta_2)} \quad (2.5)$$

$$\Psi = AW_t e^{ik_t(x\sin\theta_t - y\cos\theta_t)}$$

where W_1, W_2 and W_t are unknown reflection coefficients (the multiplier $e^{-i\omega t}$ is omitted for brevity).

After substituting (2.5) into (2.4), and solving the resulting system of linear equations we find

$$W_1 = -\frac{F_1 - L_{12}F_2 - \mu(\sin 2\theta_1 + L_{12}\sin 2\theta_2)\,\text{tg}\,2\theta_t}{F_1 - L_{12}F_2 + \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\,\text{tg}\,2\theta_t}, \quad W_2 = -\left(\frac{k_1}{k_2}\right)^2 L_{12}(1 + W_1) \quad (2.6)$$

$$W_t = \frac{1}{\cos 2\theta_t}\left(\frac{k_1}{k_t}\right)^2 [\sin 2\theta_1 + L_{12}\sin 2\theta_2 - W_1(\sin 2\theta_1 - L_{12}\sin 2\theta_2)]$$

The notation used here is as follows

$$F_1 = K_1 - 2\mu\sin^2\theta_1, \quad F_2 = K_2 - 2\mu\sin^2\theta_2, \quad L_{12} = \frac{Q + RM_1}{Q + RM_2} \quad (2.7)$$

$$K_1 = \lambda + 2\mu + QM_1, \quad K_2 = \lambda + 2\mu + QM_2,$$

At low frequencies the reflection coefficient W_2 will be proportional to ω according to (1.36) and (1.31). Thus, to the first approximation it can be assumed that only the I longitudinal and the transverse wave are reflected from the boundary.

In the case of an incidence of the wave normal to the boundary

$(\theta_1 = \theta_2 = \theta_t = 0)$

$$W_1 = -1, \qquad W_2 = 0, \qquad W_t = 0 \tag{2.8}$$

i.e. only the I longitudinal wave will be reflected.

When we let the porosity approach zero and then assume that relation (1.35) holds as well as the equalities

$$\lambda + 2\mu = \rho_s c_1^2, \ \mu = \rho_s c_t^2, \ \frac{k_1}{k_t} = \frac{c_t}{c_1} = \frac{\sin\theta_t}{\sin\theta_1}$$

where c_1 and c_t are the velocities of the longitudinal and transverse waves, we obtain the well known reflection coefficients for a continuous isotropic elastic medium [6] :

$$W_t = \frac{c_t\cos\theta_1 \mathrm{tg}^2 2\theta_t - c_1\cos\theta_t}{c_t\cos\theta_1 \mathrm{tg}^2 2\theta_t + c_1\cos\theta_t}, \qquad W_2 = 0$$

$$W_t = \frac{c_t \sin 2\theta_1}{c_1\cos 2\theta_t}\frac{2c_t\cos\theta_t}{c_t\cos\theta_1\mathrm{tg}^2 2\theta_t + c_1\cos\theta_t} \tag{2.9}$$

Assume that a II longitudinal wave is incident upon the boundary. Similarly to the above we find

$$W_1 = -\frac{1}{L_{12}}\left(\frac{k_2}{k_1}\right)^2(1 + W_2)$$

$$W_2 = -\frac{F_1 - L_{12}F_2 + \mu(\sin 2\theta_1 + L_{12}\sin 2\theta_2)\,\mathrm{tg}2\theta_t}{F_1 - L_{12}F_2 + \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\,\mathrm{tg}2\theta_t} \tag{2.10}$$

$$W_t = \frac{1}{L_{12}\cos 2\theta_t}\left(\frac{k_2}{k_t}\right)^2[\sin 2\theta_1 + L_{12}\sin 2\theta_2 + W_2(\sin 2\theta_1 - L_{12}\sin 2\theta_2)]$$

For the case of a transverse wave incident upon the boundary we have (2.11)

$$W_1 = -\left(\frac{k_t}{k_1}\right)^2\frac{\mu\sin 2\theta_t}{F_1 - L_{12}F_2}(1 - W_t), \qquad W_2 = L_{12}\left(\frac{k_t}{k_2}\right)^2\frac{\mu\sin 2\theta_t}{F_1 - L_{12}F_2}(1 - W_t)$$

$$W_t = -\frac{F_1 - L_{12}F_2 - \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\,\mathrm{tg}2\theta_t}{F_1 - L_{12}F_2 + \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\,\mathrm{tg}2\theta_t}$$

In the limiting case of a vanishing porosity we obtain from (2.11)

$$W_1 = -\frac{2c_1\cos\theta_t \mathrm{tg}2\theta_t}{c_t\cos\theta_1\mathrm{tg}^2 2\theta_t + c_1\cos\theta_t}, \ W_2 = 0, \ W_t = \frac{c_t\cos\theta_1\mathrm{tg}^2 2\theta_t - c_1\cos\theta_t}{c_t\cos\theta_1\mathrm{tg}^2 2\theta_t + c_1\cos\theta_t} \tag{2.12}$$

which coincides with the well known expressions for the reflection coefficients in continuous elastic media [6].

At the conclusion of this section let us mention the case of complete internal reflection.

We shall neglect the viscosity and assume that the velocities of the two longitudinal and the transverse waves satisfy the inequalities

$$c_1 > c_2 > c_t \tag{2.13}$$

If the transverse wave hits the boundary, complete internal reflection will take place when

$$\sin\theta_t > \frac{c_t}{c_2} \tag{2.14}$$

The angles θ_1 and θ_2 to be determined from the relations

$$\sin\theta_1 = \frac{c_1}{c_t}\sin\theta_t, \qquad \sin\theta_2 = \frac{c_2}{c_t}\sin\theta_t \tag{2.15}$$

then turn out to be complex, and the I longitudinal and the transverse waves are nonhomogeneous waves, whose amplitudes decrease exponentially with distance from the boundary. In the case of the incidence of a II longitudinal wave upon the boundary at an angle θ_2, which satisfies the condition

$$\sin\theta_2 > \frac{c_2}{c_1} \tag{2.16}$$

the angle θ_1 will be complex and the I longitudinal wave will be nonhomogeneous.

3. Surface waves. We shall study the surface waves as a degenerate case of the reflection of plane waves. To this end it is necessary, as is well known [6], to let the amplitude of the incident wave approach zero and the reflection coefficients approach infinity so that the amplitudes of the reflected waves remain finite. Thus we obtain a wave process that propagates along the boundary without an incident wave, i.e. we shall have the case of a surface wave. According to (2.6) the reflection coefficients approach infinity under the condition

$$(F_1 - L_{12}F_2)\cos 2\theta_t + \mu(\sin 2\theta_1 - L_{12}\sin 2\theta_2)\sin 2\theta_t = 0 \tag{3.1}$$

At the outset we neglect the viscosity and thus we assume all coefficients to be real. We introduce the notation

$$S = \sin^2\theta_t, \qquad q_1 = \left(\frac{c_t}{c_1}\right)^2, \qquad q_2 = \left(\frac{c_t}{c_2}\right)^2 \tag{3.2}$$

Using the relations $(\sin\theta_1)/c_1 = (\sin\theta_2)/c_2 = (\sin\theta_t)/c_t$, we reduce equation (3.1) to

$$(1-2S)\left[K_1-2\mu\frac{S}{q_1}-L_{12}\left(K_2-2\mu\frac{S}{q_2}\right)\right]+$$
$$+4\mu S\left(\frac{1}{q_1}\sqrt{q_1-S}-\frac{L_{12}}{q_2}\sqrt{q_2-S}\right)\sqrt{1-S}=0 \qquad (3.3)$$

Let us assume that condition (2.13) holds. It is necessary for the existence of surface waves that equation (3.3) has a root $S > 1$. There the sines of the angles θ_1, θ_2, and θ_t will be greater than unity, and the cosines purely imaginary, and consequently, the surface wave will damp out exponentially with distance from the boundary. Along the boundary it propagates without damping with a phase velocity

$$v=\frac{c_t}{\sqrt{S}} \qquad (3.4)$$

For $S \to \infty$, the left-hand side of equation (3.3) reduces to

$$-2S\,[K_1-\mu-L_{12}(K_2-\mu)] \qquad (3.5)$$

and for $S = 1$ it is equal to

$$-\left[K_1-\frac{2\mu}{q_1}-L_{12}\left(K_2-\frac{2\mu}{q_2}\right)\right] \qquad (3.6$$

Thus equation (3.3) has at least one root $S > 1$, when expressions (3.5) and (3.6) have opposite signs. If viscosity is taken into account, then the corresponding root of equation (3.6) will be complex and consequently also the phase velocity (3.4) will be complex. This indicates the presence of damping of the surface wave in the direction parallel to the boundary, which is caused by the dissipation of its energy due to viscosity.

In the limiting case of vanishing porosity, equation (3.3) goes over into the well known equation describing the propagation of a surface wave at the interface between a continuous elastic medium and vacuum [6]:

$$(1-2S)^2+4S\sqrt{q_1-S}\;\sqrt{1-S}=0 \qquad (3.7)$$

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